

5. Exponential Integral and Related Functions

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5. Exponential Integral and Related Functions

Mathematical Properties

5.1. Exponential Integral

Definitions

$$5.1.1 \quad E_1(z) = \int_z^{\infty} \frac{e^{-t}}{t} dt \quad (|\arg z| < \pi)$$

$$5.1.2 \quad \text{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt = \int_{-\infty}^x \frac{e^t}{t} dt \quad (x > 0)$$

$$5.1.3 \quad \text{li}(x) = \int_0^x \frac{dt}{\ln t} = \text{Ei}(\ln x) \quad (x > 1)$$

5.1.4

$$E_n(z) = \int_1^{\infty} \frac{e^{-zt}}{t^n} dt \quad (n=0, 1, 2, \dots; \Re z > 0)$$

5.1.5

$$\alpha_n(z) = \int_1^{\infty} t^n e^{-zt} dt \quad (n=0, 1, 2, \dots; \Re z > 0)$$

$$5.1.6 \quad \beta_n(z) = \int_{-1}^1 t^n e^{-zt} dt \quad (n=0, 1, 2, \dots)$$

In 5.1.1 it is assumed that the path of integration excludes the origin and does not cross the negative real axis.

Analytic continuation of the functions in 5.1.1, 5.1.2, and 5.1.4 for $n > 0$ yields multi-valued functions with branch points at $z=0$ and $z=\infty$.³ They are single-valued functions in the z -plane cut along the negative real axis.⁴ The function $\text{li}(z)$, the logarithmic integral, has an additional branch point at $z=1$.

Interrelations

5.1.7

$$E_1(-x \pm i0) = -\text{Ei}(x) \mp i\pi, \\ -\text{Ei}(x) = \frac{1}{2}[E_1(-x+i0) + E_1(-x-i0)] \quad (x > 0)$$

³ Some authors [5.14], [5.16] use the entire function $\int_0^z (1-e^{-t})dt/t$ as the basic function and denote it by $\text{Ein}(z)$. We have $\text{Ein}(z) = E_1(z) + \ln z + \gamma$.

⁴ Various authors define the integral $\int_{-\infty}^z (e^t/t)dt$ in the z -plane cut along the positive real axis and denote it also by $\text{Ei}(z)$. For $z=x > 0$ additional notations such as $\overline{\text{Ei}}(x)$ (e.g., in [5.10], [5.25]), $E^*(x)$ (in [5.2]), $\text{Ei}^*(x)$ (in [5.6]) are then used to designate the principal value of the integral. Correspondingly, $E_1(x)$ is often denoted by $-\text{Ei}(-x)$.

Explicit Expressions for $\alpha_n(z)$ and $\beta_n(z)$

$$5.1.8 \quad \alpha_n(z) = n! z^{-n-1} e^{-z} (1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!})$$

5.1.9

$$\beta_n(z) = n! z^{-n-1} \{ e^z [1 - z + \frac{z^2}{2!} - \dots + (-1)^n \frac{z^n}{n!}] - e^{-z} (1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!}) \}$$

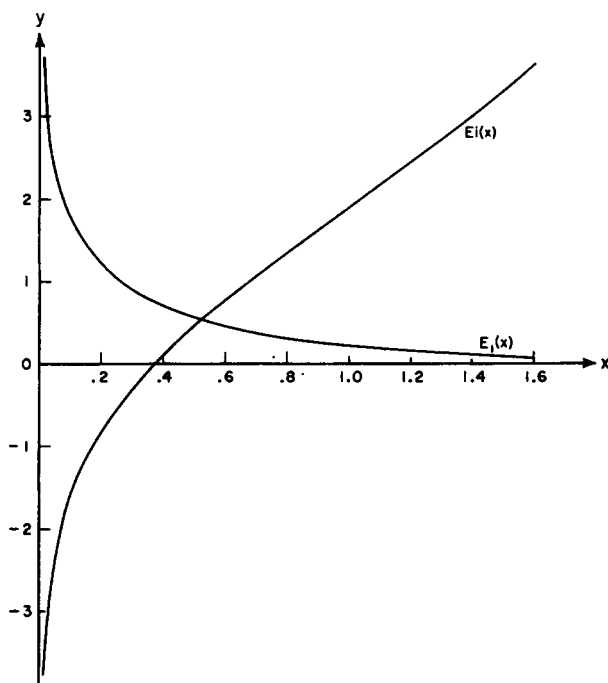


FIGURE 5.1. $y = \text{Ei}(x)$ and $y = E_1(x)$.

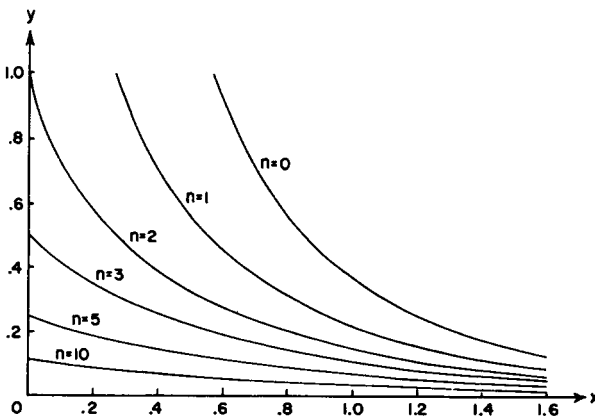


FIGURE 5.2. $y = E_n(x)$
 $n=0, 1, 2, 3, 5, 10$

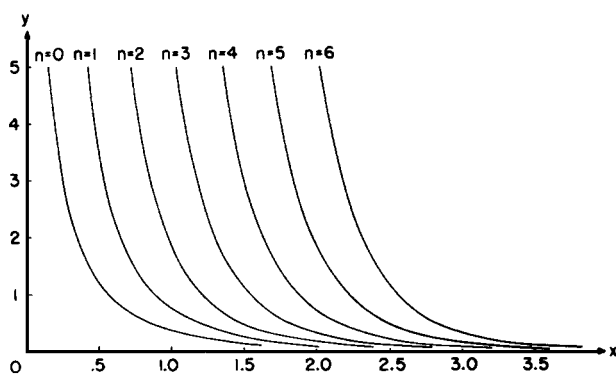


FIGURE 5.3. $y = \alpha_n(x)$
 $n = 0(1)6$

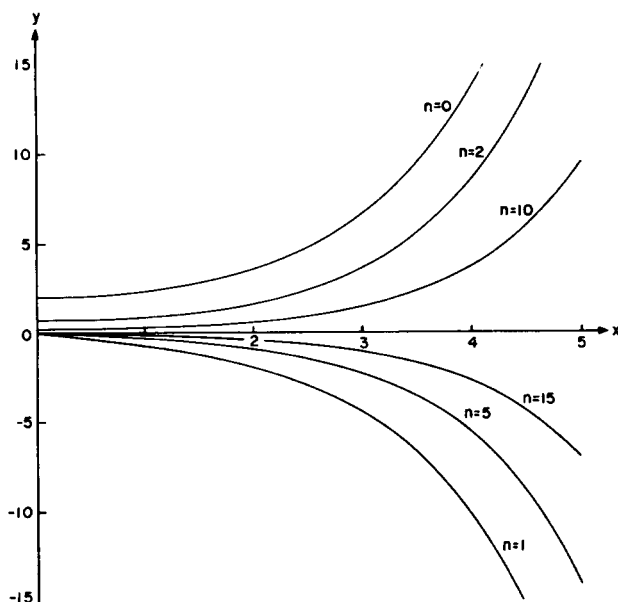


FIGURE 5.4. $y = \beta_n(x)$
 $n = 0, 1, 2, 5, 10, 15$

Series Expansions

$$5.1.10 \quad \text{Ei}(x) = \gamma + \ln x + \sum_{n=1}^{\infty} \frac{x^n}{nn!} \quad (x > 0)$$

5.1.11

$$E_1(z) = -\gamma - \ln z - \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{nn!} \quad (|\arg z| < \pi)$$

5.1.12

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!} [-\ln z + \psi(n)] - \sum_{\substack{m=0 \\ m \neq n-1}}^{\infty} \frac{(-z)^m}{(m-n+1)m!} \quad (|\arg z| < \pi)$$

$$\psi(1) = -\gamma, \quad \psi(n) = -\gamma + \sum_{m=1}^{n-1} \frac{1}{m} \quad (n > 1)$$

$\gamma = .57721\ 56649 \dots$ is Euler's constant.

Symmetry Relation

$$5.1.13 \quad E_n(\bar{z}) = \overline{E_n(z)}$$

Recurrence Relations

5.1.14

$$E_{n+1}(z) = \frac{1}{n} [e^{-z} - z E_n(z)] \quad (n = 1, 2, 3, \dots)$$

$$5.1.15 \quad z \alpha_n(z) = e^{-z} + n \alpha_{n-1}(z) \quad (n = 1, 2, 3, \dots)$$

5.1.16

$$z \beta_n(z) = (-1)^n e^z - e^{-z} + n \beta_{n-1}(z) \quad (n = 1, 2, 3, \dots)$$

Inequalities [5.8], [5.4]

5.1.17

$$\frac{n-1}{n} E_n(x) < E_{n+1}(x) < E_n(x) \quad (x > 0; n = 1, 2, 3, \dots)$$

5.1.18

$$E_n^2(x) < E_{n-1}(x) E_{n+1}(x) \quad (x > 0; n = 1, 2, 3, \dots)$$

5.1.19

$$\frac{1}{x+n} < e^x E_n(x) \leq \frac{1}{x+n-1} \quad (x > 0; n = 1, 2, 3, \dots)$$

5.1.20

$$\frac{1}{2} \ln \left(1 + \frac{2}{x} \right) < e^x E_1(x) < \ln \left(1 + \frac{1}{x} \right) \quad (x > 0)$$

5.1.21

$$\frac{d}{dx} \left[\frac{E_n(x)}{E_{n-1}(x)} \right] > 0 \quad (x > 0; n = 1, 2, 3, \dots)$$

Continued Fraction

5.1.22

$$E_n(z) = e^{-z} \left(\frac{1}{z+1} + \frac{n}{z+1} \frac{1}{z+1} \frac{n+1}{z+1} \frac{2}{z+1} \dots \right) \quad (|\arg z| < \pi)$$

Special Values

$$5.1.23 \quad E_n(0) = \frac{1}{n-1} \quad (n > 1)$$

$$5.1.24 \quad E_0(z) = \frac{e^{-z}}{z}$$

$$5.1.25 \quad \alpha_0(z) = \frac{e^{-z}}{z}, \quad \beta_0(z) = \frac{2}{z} \sinh z$$

Derivatives

$$5.1.26 \quad \frac{dE_n(z)}{dz} = -E_{n-1}(z) \quad (n=1, 2, 3, \dots)$$

5.1.27

$$\frac{d^n}{dz^n} [e^z E_1(z)] = \frac{d^{n-1}}{dz^{n-1}} [e^z E_1(z)] + \frac{(-1)^n (n-1)!}{z^n} \quad (n=1, 2, 3, \dots)$$

Definite and Indefinite Integrals

(For more extensive tables of integrals see [5.3], [5.6], [5.11], [5.12], [5.13]. For integrals involving $E_n(x)$ see [5.9].)

$$5.1.28 \quad \int_0^\infty \frac{e^{-at}}{b+t} dt = e^{ab} E_1(ab)$$

5.1.29

$$\int_0^\infty \frac{e^{iat}}{b+t} dt = e^{-iab} E_1(-iab) \quad (a>0, b>0)$$

5.1.30

$$\int_0^\infty \frac{t+ib}{t^2+b^2} e^{iat} dt = e^{ab} E_1(ab) \quad (a>0, b>0)$$

5.1.31

$$\int_0^\infty \frac{t+ib}{t^2+b^2} e^{iat} dt = e^{-ab} (-\text{Ei}(ab) + i\pi) \quad (a>0, b>0)$$

$$5.1.32 \quad \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \ln \frac{b}{a}$$

$$5.1.33 \quad \int_0^\infty E_1(t) dt = 2 \ln 2$$

5.1.34

$$\int_0^\infty e^{-at} E_n(t) dt = \frac{(-1)^{n-1}}{a^n} \left[\ln(1+a) + \sum_{k=1}^{n-1} \frac{(-1)^k a^k}{k} \right] \quad (a>-1)$$

5.1.35

$$\int_0^1 \frac{e^{at} \sin bt}{t} dt = \pi - \arctan \frac{b}{a} + \mathcal{J} E_1(-a+ib) \quad (a>0, b>0)$$

5.1.36

$$\int_0^1 \frac{e^{-at} \sin bt}{t} dt = \arctan \frac{b}{a} + \mathcal{J} E_1(a+ib) \quad (a>0, b \text{ real})$$

5.1.37

$$\int_0^1 \frac{e^{at} (1 - \cos bt)}{t} dt = \frac{1}{2} \ln \left(1 + \frac{b^2}{a^2} \right) + \text{Ei}(a) + \mathcal{J} E_1(-a+ib) \quad (a>0, b \text{ real})$$

5.1.38

$$\int_0^1 \frac{e^{-at} (1 - \cos bt)}{t} dt = \frac{1}{2} \ln \left(1 + \frac{b^2}{a^2} \right) - E_1(a) + \mathcal{J} E_1(a+ib) \quad (a>0, b \text{ real})$$

$$5.1.39 \quad \int_0^z \frac{1 - e^{-t}}{t} dt = E_1(z) + \ln z + \gamma$$

$$5.1.40 \quad \int_0^x \frac{e^t - 1}{t} dt = \text{Ei}(x) - \ln x - \gamma \quad (x>0)$$

5.1.41

$$\int \frac{e^{iz}}{a^2 + x^2} dx = \frac{i}{2a} [e^{-a} E_1(-a-ix) - e^a E_1(a-ix)] + \text{const.}$$

5.1.42

$$\int \frac{x e^{iz}}{a^2 + x^2} dx = -\frac{1}{2} [e^{-a} E_1(-a-ix) + e^a E_1(a-ix)] + \text{const.}$$

5.1.43

$$\int \frac{e^x}{a^2 + x^2} dx = -\frac{1}{a} \mathcal{J} (e^{ia} E_1(-x+ia)) + \text{const.} \quad (a>0)$$

5.1.44

$$\int \frac{x e^x}{a^2 + x^2} dx = -\mathcal{J} (e^{ia} E_1(-x+ia)) + \text{const.} \quad (a>0)$$

Relation to Incomplete Gamma Function (see 6.5)

$$5.1.45 \quad E_n(z) = z^{n-1} \Gamma(1-n, z)$$

$$5.1.46 \quad \alpha_n(z) = z^{-n-1} \Gamma(n+1, z)$$

$$5.1.47 \quad \beta_n(z) = z^{-n-1} [\Gamma(n+1, -z) - \Gamma(n+1, z)]$$

Relation to Spherical Bessel Functions (see 10.2)

$$5.1.48 \quad \alpha_0(z) = \sqrt{\frac{2}{\pi z}} K_{\frac{1}{2}}(z), \quad \beta_0(z) = \sqrt{\frac{2\pi}{z}} I_{\frac{1}{2}}(z)$$

$$5.1.49 \quad \alpha_1(z) = \sqrt{\frac{2}{\pi z}} K_{\frac{3}{2}}(z), \quad \beta_1(z) = -\sqrt{\frac{2\pi}{z}} I_{\frac{3}{2}}(z)$$

Number-Theoretic Significance of $\text{li}(x)$

(Assuming Riemann's hypothesis that all non-real zeros of $\zeta(z)$ have a real part of $\frac{1}{2}$)

$$5.1.50 \quad \text{li}(x) - \pi(x) = O(\sqrt{x} \ln x) \quad (x \rightarrow \infty)$$

$\pi(x)$ is the number of primes less than or equal to x .

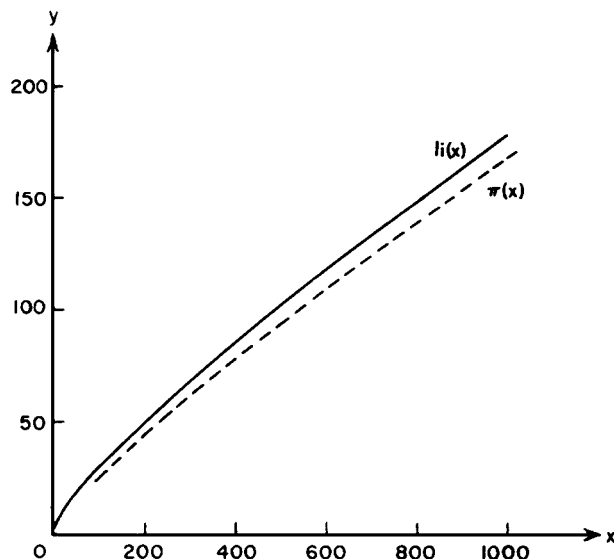


FIGURE 5.5. $y = \text{li}(x)$ and $y = \pi(x)$

Asymptotic Expansion

$$5.1.51 \quad E_n(z) \sim \frac{e^{-z}}{z} \left\{ 1 - \frac{n}{z} + \frac{n(n+1)}{z^2} - \frac{n(n+1)(n+2)}{z^3} + \dots \right\} \quad (|\arg z| < \frac{3}{2}\pi)$$

Representation of $E_n(x)$ for Large n

$$5.1.52 \quad E_n(x) = \frac{e^{-x}}{x+n} \left\{ 1 + \frac{n}{(x+n)^2} + \frac{n(n-2x)}{(x+n)^4} + \frac{n(6x^2 - 8nx + n^2)}{(x+n)^6} + R(n, x) \right\} \\ - .36n^{-4} \leq R(n, x) \leq \left(1 + \frac{1}{x+n-1} \right) n^{-4} \quad (x > 0)$$

Polynomial and Rational Approximations⁶

$$5.1.53 \quad 0 \leq x \leq 1 \\ E_1(x) + \ln x = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \epsilon(x) \\ |\epsilon(x)| < 2 \times 10^{-7}$$

⁶ The approximation 5.1.53 is from E. E. Allen, Note 169, MTAC 8, 240 (1954); approximations 5.1.54 and 5.1.56 are from C. Hastings, Jr., Approximations for digital computers, Princeton Univ. Press, Princeton, N.J., 1955; approximation 5.1.55 is from C. Hastings, Jr., Note 143, MTAC 7, 68 (1953) (with permission).

$$\begin{aligned} a_0 &= -.57721 \ 566 & a_3 &= .05519 \ 968 \\ a_1 &= .99999 \ 193 & a_4 &= -.00976 \ 004 \\ a_2 &= -.24991 \ 055 & a_5 &= .00107 \ 857 \end{aligned}$$

$$5.1.54 \quad 1 \leq x < \infty$$

$$xe^xE_1(x) = \frac{x^2 + a_1x + a_2}{x^2 + b_1x + b_2} + \epsilon(x)$$

$$|\epsilon(x)| < 5 \times 10^{-5}$$

$$\begin{aligned} a_1 &= 2.334733 & b_1 &= 3.330657 \\ a_2 &= .250621 & b_2 &= 1.681534 \end{aligned}$$

$$5.1.55 \quad 10 \leq x < \infty$$

$$xe^xE_1(x) = \frac{x^2 + a_1x + a_2}{x^2 + b_1x + b_2} + \epsilon(x)$$

$$|\epsilon(x)| < 10^{-7}$$

$$\begin{aligned} a_1 &= 4.03640 & b_1 &= 5.03637 \\ a_2 &= 1.15198 & b_2 &= 4.19160 \end{aligned}$$

$$5.1.56 \quad 1 \leq x < \infty$$

$$xe^xE_1(x) = \frac{x^4 + a_1x^3 + a_2x^2 + a_3x + a_4}{x^4 + b_1x^3 + b_2x^2 + b_3x + b_4} + \epsilon(x)$$

$$|\epsilon(x)| < 2 \times 10^{-8}$$

$$\begin{aligned} a_1 &= 8.57332 \ 87401 & b_1 &= 9.57332 \ 23454 \\ a_2 &= 18.05901 \ 69730 & b_2 &= 25.63295 \ 61486 \\ a_3 &= 8.63476 \ 08925 & b_3 &= 21.09965 \ 30827 \\ a_4 &= .26777 \ 37343 & b_4 &= 3.95849 \ 69228 \end{aligned}$$

5.2. Sine and Cosine Integrals

Definitions

$$5.2.1 \quad \text{Si}(z) = \int_0^z \frac{\sin t}{t} dt$$

$$5.2.2^6 \quad \text{Ci}(z) = \gamma + \ln z + \int_0^z \frac{\cos t - 1}{t} dt \quad (|\arg z| < \pi)$$

$$5.2.3^7 \quad \text{Shi}(z) = \int_0^z \frac{\sinh t}{t} dt$$

$$5.2.4^7 \quad \text{Chi}(z) = \gamma + \ln z + \int_0^z \frac{\cosh t - 1}{t} dt \quad (|\arg z| < \pi)$$

⁶ Some authors [5.14], [5.16] use the entire function $\int_0^z (1 - \cos t) dt/t$ as the basic function and denote it by $\text{Cin}(z)$. We have

$$\text{Cin}(z) = -\text{Ci}(z) + \ln z + \gamma.$$

⁷ The notations $\text{Sih}(z) = \int_0^z \sinh t dt/t$, $\text{Cinh}(z) = \int_0^z (\cosh t - 1) dt/t$ have also been proposed [5.14.]

$$5.2.5 \quad \text{si}(z) = \text{Si}(z) - \frac{\pi}{2}$$

Auxiliary Functions

$$5.2.6 \quad f(z) = \text{Ci}(z) \sin z - \text{si}(z) \cos z$$

$$5.2.7 \quad g(z) = -\text{Ci}(z) \cos z - \text{si}(z) \sin z$$

Sine and Cosine Integrals in Terms of Auxiliary Functions

$$5.2.8 \quad \text{Si}(z) = \frac{\pi}{2} - f(z) \cos z - g(z) \sin z$$

$$5.2.9 \quad \text{Ci}(z) = f(z) \sin z - g(z) \cos z$$

Integral Representations

$$5.2.10 \quad \text{si}(z) = - \int_0^{\frac{\pi}{2}} e^{-z \cos t} \cos(z \sin t) dt$$

$$5.2.11 \quad \text{Ci}(z) + E_1(z) = \int_0^{\frac{\pi}{2}} e^{-z \cos t} \sin(z \sin t) dt$$

$$5.2.12 \quad f(z) = \int_0^{\infty} \frac{\sin t}{t+z} dt = \int_0^{\infty} \frac{e^{-zt}}{t^2+1} dt \quad (\Re z > 0)$$

$$5.2.13 \quad g(z) = \int_0^{\infty} \frac{\cos t}{t+z} dt = \int_0^{\infty} \frac{te^{-zt}}{t^2+1} dt \quad (\Re z > 0)$$

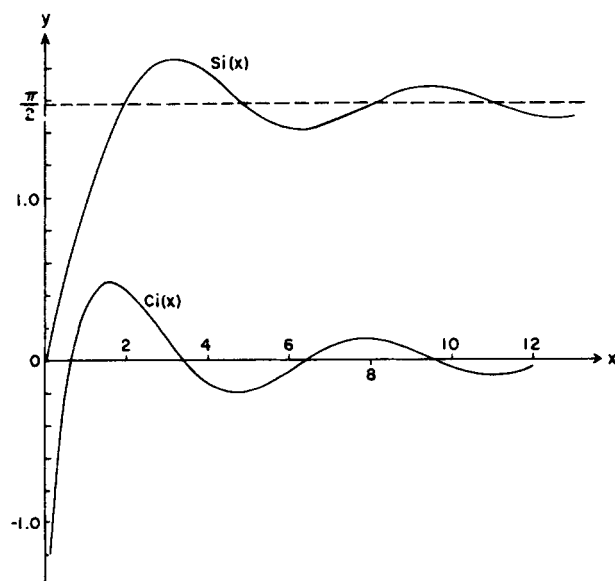


FIGURE 5.6. $y = \text{Si}(x)$ and $y = \text{Ci}(x)$

Series Expansions

$$5.2.14 \quad \text{Si}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)(2n+1)!}$$

$$5.2.15 \quad \text{Si}(z) = \pi \sum_{n=0}^{\infty} J_{n+1}^2\left(\frac{z}{2}\right)$$

$$5.2.16 \quad \text{Ci}(z) = \gamma + \ln z + \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{2n(2n)!}$$

$$5.2.17 \quad \text{Shi}(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)(2n+1)!}$$

$$5.2.18 \quad \text{Chi}(z) = \gamma + \ln z + \sum_{n=1}^{\infty} \frac{z^{2n}}{2n(2n)!}$$

Symmetry Relations

$$5.2.19 \quad \text{Si}(-z) = -\text{Si}(z), \quad \text{Si}(\bar{z}) = \overline{\text{Si}(z)}$$

$$5.2.20$$

$$\text{Ci}(-z) = \text{Ci}(z) - i\pi \quad (0 < \arg z < \pi)$$

$$\text{Ci}(\bar{z}) = \overline{\text{Ci}(z)}$$

Relation to Exponential Integral

$$5.2.21$$

$$\text{Si}(z) = \frac{1}{2i} [E_1(iz) - E_1(-iz)] + \frac{\pi}{2} \quad (|\arg z| < \frac{\pi}{2})$$

$$5.2.22 \quad \text{Si}(ix) = \frac{i}{2} [\text{Ei}(x) + E_1(x)] \quad (x > 0)$$

$$5.2.23$$

$$\text{Ci}(z) = -\frac{1}{2} [E_1(iz) + E_1(-iz)] \quad (|\arg z| < \frac{\pi}{2})$$

$$5.2.24 \quad \text{Ci}(ix) = \frac{1}{2} [\text{Ei}(x) - E_1(x)] + i\frac{\pi}{2} \quad (x > 0)$$

Value at Infinity

$$5.2.25 \quad \lim_{x \rightarrow \infty} \text{Si}(x) = \frac{\pi}{2}$$

Integrals

(For more extensive tables of integrals see [5.3], [5.6], [5.11], [5.12], [5.13].)

$$5.2.26 \quad \int_z^{\infty} \frac{\sin t}{t} dt = -\text{si}(z) \quad (|\arg z| < \pi)$$

$$5.2.27 \quad \int_z^{\infty} \frac{\cos t}{t} dt = -\text{Ci}(z) \quad (|\arg z| < \pi)$$

$$5.2.28 \quad \int_0^{\infty} e^{-at} \text{Ci}(t) dt = -\frac{1}{2a} \ln(1+a^2) \quad (\Re a > 0)^*$$

$$5.2.29 \quad \int_0^{\infty} e^{-at} \text{si}(t) dt = -\frac{1}{a} \arctan a \quad (\Re a > 0)$$

$$5.2.30 \quad \int_0^{\infty} \cos t \text{Ci}(t) dt = \int_0^{\infty} \sin t \text{si}(t) dt = -\frac{\pi}{4}$$

*See page 11.

$$5.2.31 \quad \int_0^{\infty} \text{Ci}^2(t) dt = \int_0^{\infty} \text{si}^2(t) dt = \frac{\pi}{2}$$

$$5.2.32^* \quad \int_0^{\infty} \text{Ci}(t) \text{si}(t) dt = \ln 2$$

5.2.33

$$\int_0^1 \frac{(1-e^{-at}) \cos bt}{t} dt = \frac{1}{2} \ln \left(1 + \frac{a^2}{b^2} \right) + \text{Ci}(b) \\ + \mathcal{R} E_1(a+ib) \quad (a \text{ real}, b > 0)$$

Asymptotic Expansions

5.2.34

$$f(z) \sim \frac{1}{z} \left(1 - \frac{2!}{z^2} + \frac{4!}{z^4} - \frac{6!}{z^6} + \dots \right) \quad (|\arg z| < \pi)$$

5.2.35

$$g(z) \sim \frac{1}{z^2} \left(1 - \frac{3!}{z^2} + \frac{5!}{z^4} - \frac{7!}{z^6} + \dots \right) \quad (|\arg z| < \pi)$$

Rational Approximations⁸

5.2.36

$$1 \leq x < \infty$$

$$f(x) = \frac{1}{x} \left(\frac{x^4 + a_1 x^2 + a_2}{x^4 + b_1 x^2 + b_2} \right) + \epsilon(x)$$

$$|\epsilon(x)| < 2 \times 10^{-4}$$

$$a_1 = 7.241163 \quad b_1 = 9.068580$$

$$a_2 = 2.463936 \quad b_2 = 7.157433$$

5.2.37

$$1 \leq x < \infty$$

$$g(x) = \frac{1}{x^2} \left(\frac{x^4 + a_1 x^2 + a_2}{x^4 + b_1 x^2 + b_2} \right) + \epsilon(x)$$

$$|\epsilon(x)| < 10^{-4}$$

$$a_1 = 7.547478 \quad b_1 = 12.723684 \quad *$$

$$a_2 = 1.564072 \quad b_2 = 15.723606 \quad *$$

5.2.38

$$1 \leq x < \infty$$

$$f(x) = \frac{1}{x} \left(\frac{x^8 + a_1 x^6 + a_2 x^4 + a_3 x^2 + a_4}{x^8 + b_1 x^6 + b_2 x^4 + b_3 x^2 + b_4} \right) + \epsilon(x)$$

$$|\epsilon(x)| < 5 \times 10^{-7}$$

$$a_1 = 38.027264 \quad b_1 = 40.021433$$

$$a_2 = 265.187033 \quad b_2 = 322.624911$$

$$a_3 = 335.677320 \quad b_3 = 570.236280$$

$$a_4 = 38.102495 \quad b_4 = 157.105423$$

5.2.39

$$1 \leq x < \infty$$

$$g(x) = \frac{1}{x^2} \left(\frac{x^8 + a_1 x^6 + a_2 x^4 + a_3 x^2 + a_4}{x^8 + b_1 x^6 + b_2 x^4 + b_3 x^2 + b_4} \right) + \epsilon(x)$$

$$|\epsilon(x)| < 3 \times 10^{-7}$$

$$a_1 = 42.242855 \quad b_1 = 48.196927$$

$$a_2 = 302.757865 \quad b_2 = 482.485984$$

$$a_3 = 352.018498 \quad b_3 = 1114.978885$$

$$a_4 = 21.821899 \quad b_4 = 449.690326$$

Numerical Methods**5.3. Use and Extension of the Tables****Example 1.** Compute Ci (.25) to 5D.

From Tables 5.1 and 4.2 we have

$$\frac{\text{Ci}(.25) - \ln(.25) - \gamma}{(.25)^2} = -.249350,$$

$$\text{Ci}(.25) = (.25)^2(-.249350) + (-1.38629) \\ + .577216 = -.82466.$$

Example 2. Compute Ei (8) to 5S.From Table 5.1 we have $xe^{-x}\text{Ei}(x) = 1.18185$ for $x=8$. From Table 4.4, $e^8 = 2.98096 \times 10^3$. Thus $\text{Ei}(8) = 440.38$.**Example 3.** Compute Si (20) to 5D.Since $1/20 = .05$ from Table 5.2 we find $f(20) = .049757$, $g(20) = .002464$. From Table 4.8, $\sin 20 = .912945$, $\cos 20 = .408082$. Using 5.2.8

$$\text{Si}(20) = \frac{\pi}{2} - f(20) \cos 20 - g(20) \sin 20 \\ = 1.570796 - .022555 = 1.54824.$$

Example 4. Compute $E_n(x)$, $n=1(1)N$, to 5S for $x=1.275$, $N=10$.If x is less than about five, the recurrence relation 5.1.14 can be used in increasing order of n without serious loss of accuracy.By quadratic interpolation in Table 5.1 we get $E_1(1.275) = .1408099$, and from Table 4.4, $e^{-1.275} = .2794310$. The recurrence formula 5.1.14 then yields⁸See page II.⁸From C. Hastings, Jr., Approximations for digital computers, Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

n	$E_n(1.275)$	$E_n(1.275)$
1	.1408099	6 .0430168
2	.0998984	7 .0374307
3	.0760303	8 .0331009
4	.0608307	9 .0296534
5	.0504679	10 .0268469

Interpolating directly in Table 5.4 for $n=10$ we get $E_{10}(1.275)=.0268470$ as a check.

Example 5. Compute $E_n(x)$, $n=1(1)N$, to 5S for $x=10$, $N=10$.

If, as in this example, x is appreciably larger than five and $N \leq x$, then the recurrence relation 5.1.14 may be safely used in decreasing order of n ([5.5]). From Table 5.5 for $x^{-1}=.1$ we get $(x+10)e^x E_{10}(x)=1.02436$ so that $E_{10}(10)=2.32529 \times 10^{-6}$. Using this as the initial value we obtain column (2).

n	$10^6 E_n(10)$ (1)	$10^6 E_n(10)$ (2)
1	.41570	.41570
2	.38300	.38302
3	.35500	.35488
4	.33000	.33041
5	.31000	.30898
6	.28800	.29005
7	.27667	.27325
8	.25333	.25822
9	.25084	.24472
10	.22573	.23253

From Table 5.2 we get $xe^x E_1(x)=.915633$ so that $E_1(10)=4.15697 \times 10^{-6}$ as a check. Forward recurrence starting with $E_1(10)=4.1570 \times 10^{-6}$ yields the values in column (1). The underlined figures are in error.

Example 6. Compute $E_n(x)$, $n=1(1)N$, to 5S for $x=12.3$, $N=20$.

If N is appreciably larger than x , and x appreciably larger than five, then the recurrence relation 5.1.14 should be used in the backward direction to generate $E_n(x)$ for $n < n_0$, and in the forward direction to generate $E_n(x)$ for $n > n_0$, where $n_0 = \langle x \rangle$.

From 5.1.52, with $n_0=12$, $x=12.3$, we have

$$E_{n_0}(x) = \frac{e^{-12.3}}{24.3} (1 + .02032 - .00043 - .00001) = 1.91038 \times 10^{-7}.$$

Using the recurrence relation 5.1.14, as indicated, we get

n	$10^6 E_n(12.3)$	$10^6 E_n(12.3)$	n
12	.191038	.191038	12
11	.199213	.183498	13
10	.208098	.176516	14
9	.217793	.170042	15
8	.228406	.164015	16
7	.240073	.158397	17
6	.252951	.153144	18
5	.267234	.148226	19
4	.283155	.143608	20
3	.300998		
2	.321117		
1	.343953		

From Tables 5.2 and 5.5 we find $E_1(12.3)=.343953 \times 10^{-6}$, $E_{20}(12.3)=.143609 \times 10^{-6}$ as a check.

Example 7. Compute $\alpha_n(2)$ to 6S for $n=1(1)5$.

The recurrence formula 5.1.15 can be used for all $x > 0$ in increasing order of n without loss of accuracy. From 5.1.25 we have $\alpha_0(2) = \frac{1}{2} e^{-2} = .0676676$, so we get

n	$\alpha_n(2)$
0	.0676676
1	.101501
2	.169169
3	.321421
4	.710510
5	1.84394

Independent calculation with 5.1.8 yields the same result for $\alpha_5(2)$.

The functions $\alpha_0(x)$ and $\alpha_1(x)$ can be obtained from Table 10.8 using 5.1.48, 5.1.49.

Example 8. Compute $\beta_n(x)$, $n=0(1)N$ to 6S for $x=1$, $N=5$.

Use the recurrence relation 5.1.16 in increasing order of n if

$$x > .368N + .184 \ln N + .821$$

and in decreasing order of n otherwise [5.5].

From 5.1.9 with $n=5$ we get $\beta_5(1) = -.324297$ correctly rounded to 6D. Using the recurrence formula 5.1.16 in decreasing order of n and carrying 9D we get the values in column (2).

n	$\beta_n(1)$ (1)	$\beta_n(1)$ (2)
0	2.35040 2	2.35040 2389
1	-.73575 9269	-.73575 8880
2	.87888 3849	.87888 4629
3	-.44950 9722	-.44950 7383
4	.55236 3499	.55237 2854
5	-.32434 3774	-.32429 7

Using forward recurrence instead, starting with

$\beta_0(1)=2 \sinh 1=2.350402$ and again carrying 9D, we obtain column (1). The underlined figures are in error. The above shows that three significant figures are lost in forward recurrence, whereas about three significant figures are gained in backward recurrence!

An alternative procedure is to start with an arbitrary value for n sufficiently large (see also [5.1]). To illustrate, starting with the value zero at $n=11$ we get

n	$\beta_n(1)$	n	$\beta_n(1)$
11	0.	5	-.324297
10	.280560	4	.552373
9	-.206984	3	-.449507
8	.319908	2	.878885
7	-.253812	1	-.735759
6	.404621	0	2.350402

The functions $\beta_0(x)$ and $\beta_1(x)$ can be obtained from Table 10.8 using 5.1.48, 5.1.49.

Example 9. Compute $E_1(z)$ for $z=3.2578+6.8943i$.

From Table 5.6 we have for $z_0=x_0+iy_0=3+7i$

$$z_0 e^{z_0} E_1(z_0) = .934958 + .095598i,$$

$$e^{z_0} E_1(z_0) = .059898 - .107895i.$$

From Taylor's formula with $f(z)=e^z E_1(z)$ we have

$$f(z) = f(z_0 + \Delta z) = f(z_0) + \frac{f'(z_0)}{1!} \Delta z + \frac{f''(z_0)}{2!} (\Delta z)^2 + \dots$$

with $\Delta z = z - z_0 = .2578 - .1057i$. Thus with 5.1.27 we get

k	$f^{(k)}(z_0)/k!$	$(\Delta z)^k f^{(k)}(z_0)/k!$
0	.059898 - .107895i	.059898 - .107895i
1	.008174 + .012795i	.003460 + .002435i
2	-.001859 + .000155i	-.000094 + .000110i
3	.000088 - .000212i	-.000003 - .000004i

$$f(z) = .063261 - .105354i$$

$$e^{-z} = .031510 - .022075i$$

$$E_1(z) = -.000332 - .004716i$$

Repeating the calculation with $z_0=3+6i$ and $\Delta z=.2578+.8943i$ we get the same result.

An alternative procedure is to perform bivariate interpolation in the real and imaginary parts of $ze^z E_1(z)$.

Example 10. Compute $E_1(z)$ for $z=-4.2+12.7i$.

Using the formula at the bottom of Table 5.6

$$\begin{aligned} e^z E_1(z) &\approx \frac{.711093}{-3.784225 + 12.7i} \\ &\quad + \frac{.278518}{-1.90572 + 12.7i} + \frac{.010389}{2.0900 + 12.7i} \\ &= -.0184106 - .0736698i \\ E_1(z) &\approx -1.87133 - 4.70540i. \end{aligned}$$

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